Strategy for Solving Optimization Problems with Linear Programming

Many applications involve minimizing or maximizing a certain quantity (often money) for a given scenario (usually involving restrictions). A process called *linear programming* can often be used to "optimize" such a quantity under these given "constraints".

For such a problem, first identify the quantity to be optimized. Create a function that can be used to find this quantity for any situation (this function will generally involve two variables). This function is called the *objective function*.

Next, identify any restrictions on the objective function's variables. Create inequalities to represent these restrictions. The resulting system of inequalities is known as the set of *constraints*.

Now that you have the objective function and the constraints, you can begin the linear programming process. Start by graphing the solution to the system of inequalities defined by the constraints. This solution is known as the *feasible region*. Identify all *vertices* of this region.

Evaluate the objective function at each vertex (a table helps keep this organized). If the feasible region is bounded, the minimum and maximum objective function values will be one of these values (the minimum is the least value, the maximum is the greatest). If the feasible region is unbounded, there will only be a minimum *or* maximum value. Simply test an additional point in the feasible region far from any vertex to see if the outer values get greater in the positive or negative direction.

Example

A small company produces bicycles and skateboards, and its profit for each bicycle is $35 and for each skateboard is $20. For any week, the company can produce at most 160 units (a combined total of bicycles and skateboards). Its factory can only manufacture 60 bicycles a day, but it can produce 120 skateboards daily. How many bicycles and skateboards should the company produce to maximize profit?

The information that follows shows how linear programming can be used to answer this question.

The Objective Function

The goal here is to maximize profit, so that is what the objective function should represent. From the given information, the profit for each bicycle is $35 and for each skateboard is $20. If B represents the number of bicycles produced and S the number of skateboards produced, the total profit (in dollars) could be found as $35B + 20S$. This means the objective function can be written as

$$p(B, S) = 35B + 20S,$$

where $p(B, S)$ is the profit from producing B bicycles and S skateboards.

The Constraints

The constraints are inequalities that represent the restrictions in the application. The trick is to remember that each constraining inequality must be written in terms of the same variables as the objective function, B and S in this case.

The first given restriction is that only 160 total units can be produced, although fewer would be possible. This gives the inequality $B + S \leq 160$.

Needless to say, a negative number of bicycles cannot be produced. There is also a restriction on the maximum number of bicycles that can be made, 60. This gives the inequality $0 \leq B \leq 60$.

Along the same lines, the number of skateboard is restricted to a maximum of 120, so the last constraint is $0 \leq S \leq 120$.

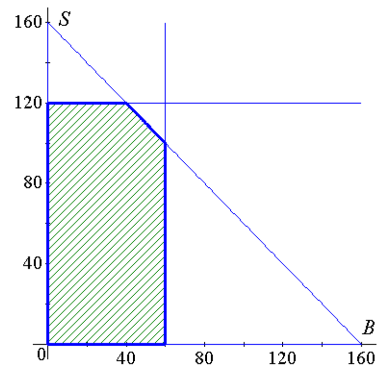
Note that these last two constraints were compound inequalities, but they each could have been listed as two separate simple inequalities (for example, $0 \leq B \leq 60$ could be $B \geq 0$ and $B \leq 60$).

The complete set of constraints can now be written as the system of inequalities

$$\begin{cases} B + S \leq 160 \\ 0 \leq B \leq 60 \\ 0 \leq S \leq 120 \end{cases}.$$

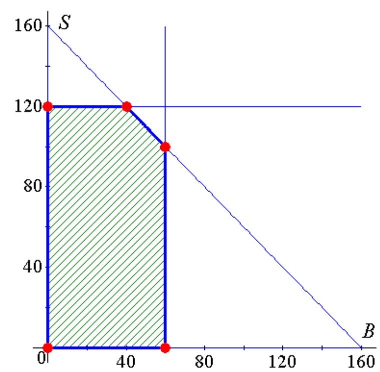
The Feasible Region

Next is to graph the system of equations to find the feasible region – this represents all of the possible combinations for *B* and *S* that are allowed to be used in the objective function. In this example, *B* is represented on the *x*-axis and *S* on the *y*-axis.



The Vertices

From the graph of the feasible region, there are five vertices (corners in the region).



Most vertices can be determined visually as the system of inequalities is graphed. If the exact location of a vertex is unclear, however, use a system of *equations* to find where the boundary lines intersect.

In this example, the vertices are (0, 0), (60, 0), (60, 100), (40, 120), and (0, 120).

Optimization

Since the feasible region is bounded (it is totally enclosed by the lines), there will be a minimum and maximum value for the objective function. The idea behind linear programming is that these values will occur at a vertex.

Set up a table and test each vertex in the objective function.

Vertex of Feasible Region	Evaluation Calculation	Objective Function Value
(0, 0)	$p(0, 0) = 35(0) + 20(0)$	$p(0, 0) = 0$
(60, 0)	$p(60, 0) = 35(60) + 20(0)$	$p(60, 0) = 2100$
(60, 100)	$p(60, 100) = 35(60) + 20(100)$	$p(60, 100) = 4100$
(40, 120)	$p(40, 120) = 35(40) + 20(120)$	$p(40, 120) = 3800$
(0, 120)	$p(0, 120) = 35(0) + 20(120)$	$p(0, 0) = 2400$

From the table, the objective function has a minimum of $p(0, 0) = 0$ and a maximum of $p(60, 100) = 4100$.

Conclusion

The original question asked to find how many bicycles and skateboards the company should produce to maximize profit? From the table, the maximum objective function value is $p(60, 100) = 4100$. This means that the maximum profit would occur when 60 bicycles and 100 skateboards are produced. Furthermore, that profit would be $4100.