A Summary of Polynomial Functions

Introduction

The purpose of this document is to summarize the information on polynomial expressions, equations, and functions as presented in class. The intent is to provide the essential concepts in an efficient fashion, "streamlining" the theory to only a few key points.

The use of a graphing calculator that can find the x-intercepts of any arbitrary function is assumed, as is the prerequisite skill of factoring polynomials.

Terminology

A polynomial is any expression that has the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 ,$$

where n (the degree of the polynomial) is a non-negative integer and a_n (the nth term coefficient) is a real number.

Throughout this document, the expression $p(x)$ will represent any generic polynomial of degree n. It is also assumed that c is any complex number (including real and pure imaginary) in the form $a + bi$, where a and b are real numbers.

The following expressions are equivalent (if one statement is found to be true, the other three are true as well):

- $x = c$ is a zero of the polynomial $p(x)$
- $x = c$ is a solution or root to the equation $p(x) = 0$
- $(x - c)$ is a factor of the polynomial $p(x)$
- if c is a real number, $(c, 0)$ is an x-intercept to the graph $y = p(x)$

Theorems and Useful Facts

While there are several theorems and corollaries involving polynomials, the following are the most useful in high school algebra.

Corollaries to the Fundamental Theorem of Algebra

- A polynomial of degree n will have exactly n complex zeros.
- A polynomial of degree n can be written as the product of exactly n linear factors. Specifically,

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &= a\left(x - c_1\right)\left(x - c_2\right)\cdots\left(x - c_{n-1}\right)\left(x - c_n\right) \end{aligned}$$

The Remainder and Factor Theorems

- If $p(x)$ is divided by $(x - c)$, the remainder will be $p(c)$.
- If $p(c) = 0$, then $(x - c)$ is a factor of $p(x)$.

Complex Zeros of a Polynomial

- If $x = a + bi$ is a zero of $p(x)$ and $b \neq 0$, then $a - bi$ is also a zero of $p(x)$. In other words, complex zeros occur as conjugate pairs.

Rational Root Theorem

- Every rational zero of $p(x)$ can be written as in the form $x = P/Q$ where P is a factor of the constant term and Q is a factor of the leading coefficient.

Technique for Finding All Zeros of a Polynomial

While the steps that following are for finding the zeros of a polynomial, the same idea can be used to find solutions/roots to polynomial equations as well as writing polynomials as a product of linear terms.

Factoring, when applicable, is the quickest way to find zeros of a polynomial. At any point in the process, factor the original or depressed polynomials as much as possible. Aside from the techniques and factoring patterns learned in beginning algebra, the following can be useful here as well:

- sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
- difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- sum of two squares: $x^2 + a^2 = (x + ai)(x - ai)$

To find the zeros of $p(x)$, you can do the following:

1. First, graph $y = p(x)$ and locate any x-intercepts for which exact values are known. These are the rational zeros of $p(x)$.
   - If the graph "bumps" the x-axis, there is a double root. The zero has a multiplicity of 2 and can be considered twice.
   - If the graph "rides" the x-axis, there is a triple root. The zero has a multiplicity of 3 and can be considered three times.
   - Ignore the irrational x-intercepts.

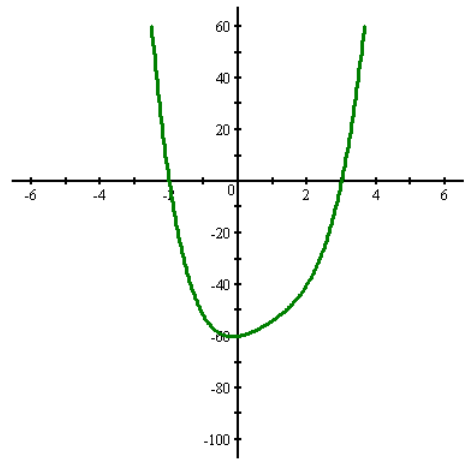
2. Second, use synthetic division with each of the rational zeros.
   - If there are multiple rational zeros, perform synthetic division on $p(x)$ with one. Then divide again using the next rational zero on the depressed polynomial resulting from the first division. Repeat as necessary.
   - Zeros with multiplicity of k can be used k times with the division.
   - Once the division results in a quadratic depressed polynomial, factor or use the quadratic formula to find the last two zeros.

3. List the zeros.
   - Double check that there are as many zeros as the degree of $p(x)$, including multiplicity.
   - Double check that the complex zeros occurred in conjugate pairs.

Example 1

Find the zeros to the polynomial $p(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$.

First, graph $y = p(x)$ to locate the two x-intercepts at $(-2, 0)$ and $(3, 0)$. These represent the two rational zeros $x = -2$ and $x = 3$.



Next, use these two zeros with synthetic division to divide out the factors $(x + 2)$ and $(x - 3)$.

−2	1	−3	6	2	−60
	↓	−2	10	−32	60
	1	−5	16	−30	0

3	1	−5	16	−30
	↓	3	−6	30
	1	−2	10	0

The division shows that the original polynomial can be written "partially" factored using the depressed polynomial as one of the factors: $p(x) = (x + 2)(x - 3)(x^2 - 2x + 10)$.

The remaining zeros are "contained" within the quadratic factor. Since it is not factorable, use the quadratic formula.

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-36}}{2} \\ &= 1 \pm 3i \end{aligned}$$

Therefore, $p(x)$ has the zeros $x = -2, 3, 1 + 3i, 1 - 3i$. Notice there are four zeros, corresponding to the degree of $p(x)$.

NOTE: $p(x)$ could be written in factored form as $p(x) = (x + 2)(x - 3)(x - 1 - 3i)(x - 1 + 3i)$.

Example 2

Find the zeros to the polynomial $p(x) = x^4 + 4x^2 - 45$.

One might try to graph $y = p(x)$ as before, but the only two x-intercepts found are irrational. Another technique is required. Notice this time that $p(x)$ is a trinomial, so look to see if it is factorable. It turns out, it is!

$$p(x) = (x^2 - 5)(x^2 + 9)$$

The first of the two factors is the difference of two squares (just not perfect squares) while the second factor is a sum of two squares. Each can be factored further to write $p(x)$ as the product of four linear factors.

$$p(x) = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)\left(x + 3i\right)\left(x - 3i\right)$$

Set each factor equal to zero and solve to find the four zeros of the polynomial,

$$x = \pm\sqrt{5}, \pm 3i\,.$$