

### Example 1

Evaluate each limit using the given table of values.

$x$	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	5.25	5.48	5.499	?	5.501	5.52	5.75

$$\lim_{x \rightarrow 2^-} f(x) = 5.5$$

$$\lim_{x \rightarrow 2^+} f(x) = 5.5$$

$$\lim_{x \rightarrow 2} f(x) = 5.5$$

Notice how the one-sided limits are equal to each other and also to the two-sided limit.

$t$	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$h(t)$	9.11	9.05	9.002	?	2.499	2.49	2.3

$$\lim_{t \rightarrow -3^-} h(t) = 9$$

$$\lim_{t \rightarrow -3^+} h(t) = 2.5$$

$$\lim_{t \rightarrow -3} h(t) = DNE$$

“DNE” means “does not exist”. Notice how the one-sided limits are not equal to each other, so the two-sided limit is non-existent since it does not approach a single value from both sides.

$\theta$	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$a(\theta)$	2.9	17.6	123.51	?	12	12	12

$$\lim_{\theta \rightarrow 0^-} a(\theta) = \infty$$

$$\lim_{\theta \rightarrow 0^+} a(\theta) = 12$$

$$\lim_{\theta \rightarrow 0} a(\theta) = DNE$$

You can also say that the one-sided limit from the left does not exist since it does not approach a single value (it is unbounded), but stating positive infinity is more descriptive.

$y$	0.9	0.99	0.999	1	1.001	1.01	1.1
$g(y)$	-2	-2.4	-2.95	?	4.6	-2.1	3.8

$$\lim_{y \rightarrow 1^-} g(y) = -2.5$$

$$\lim_{y \rightarrow 1^+} g(y) = DNE$$

$$\lim_{y \rightarrow 1} g(y) = DNE$$

The given values from the left of  $y = 1$  do not provide enough information for a confident evaluation for that one-sided limit, but  $-2.5$  is a reasonable guess. The values from the right of  $y = 1$  bounce from positive to negative, so the one-sided limit from the right does not seem to approach a single value.

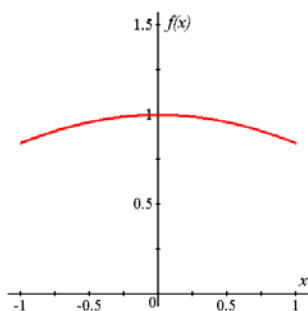
## Example 2

Suppose  $f(x) = \frac{\sin x}{x}$ . Find the limit of  $f(x)$  as  $x \rightarrow 0$ .

If you substitute  $x = 0$  into the function, the result is  $f(0) = \frac{\sin 0}{0} = \frac{0}{0}$ . Since this is undefined, the limit evaluated with a direct substitution is “indeterminate”.

If you graph  $f(x)$  in the calculator (adjust the window to get a reasonable picture near  $x = 0$ ), you can see that the function values approach  $y = 1$  as  $x$  gets closer to  $x = 0$  from either side (see graph below).

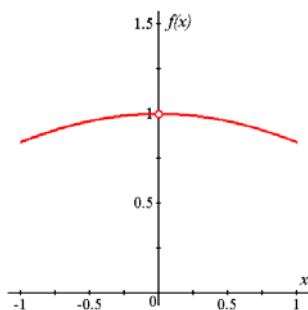
Another technique is to evaluate the function for several values to the immediate left and right of  $x = 0$ . From the table of values shown below, each of the one-sided limits appear to approach  $y = 1$  as well.



$x$	$f(x)$
-0.1	0.9983342
-0.01	0.9999833
-0.001	0.9999998
0	?

$x$	$f(x)$
0.1	0.9983342
0.01	0.9999833
0.001	0.9999998
0	?

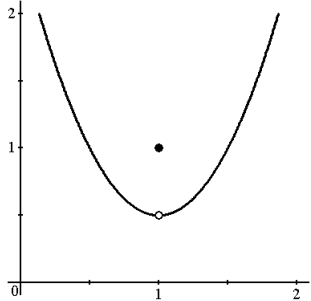
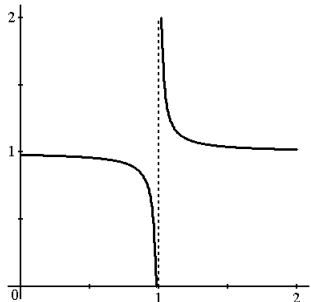
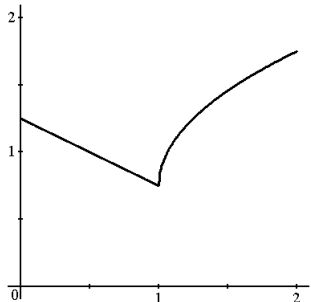
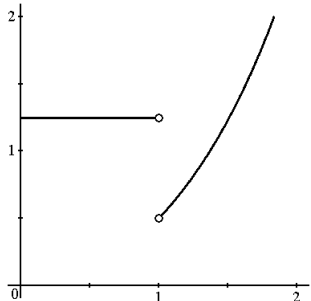
For the record, the graph shown above is a little deceptive. Since  $f(0)$  was undefined, there is no function value when  $x = 0$ . The graph really has a “hole” in the curve at the point  $(0, 1)$ , as shown below.



Notice the  $y$ -coordinate of the hole's location is also the limit value.

### Example 3

For each graph shown, determine if the limit as  $x \rightarrow 1$  exists. Explain your reasoning.

	<p>Since <math>y \rightarrow 0.5</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow 0.5</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are equal and the two-sided limit exists:</p> $\lim_{x \rightarrow 1} y(x) = 0.5$ <p>Notice that <math>y(1) = 1</math>, which is different from the limit value. Remember that the function value has no effect on the limit value.</p>
	<p>Since <math>y \rightarrow -\infty</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow \infty</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are unbounded in opposite directions and the two-sided limit does not exist:</p> $\lim_{x \rightarrow 1} y(x) = DNE$ <p>Note that you cannot describe the two-sided limit as approaching infinity since the one-sided limits were in opposite directions.</p>
	<p>Since <math>y \rightarrow 0.75</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow 0.75</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are equal and the two-sided limit exists:</p> $\lim_{x \rightarrow 1} y(x) = 0.75$ <p>Unlike the first graph, the function value here is equal to the limit value. Although the function value is unrelated to the limit value, they can be equal.</p>
	<p>Since <math>y \rightarrow 1.25</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow 0.5</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are not equal and the two-sided limit does not exist:</p> $\lim_{x \rightarrow 1} y(x) = DNE$ <p>Notice how there is a “jump” in the graph. Limits will fail to exist every time at a jump since the one-sided limits approach different values.</p>

	<p>Since <math>y \rightarrow 1</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow 1</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are equal and the two-sided limit exists:</p> $\lim_{x \rightarrow 1} y(x) = 1$ <p>Notice that <math>y(1)</math> is undefined, unlike the first and third graphs. Since the function and limit values are independent of each other, the function does not have to be defined for the limit to exist.</p>
	<p>Since <math>y \rightarrow \infty</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow \infty</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are unbounded in the same direction. The two-sided limit does not exist, but can be described as:</p> $\lim_{x \rightarrow 1} y(x) = \infty$ <p>Compare this to the second graph, which was unbounded in opposite directions.</p>
	<p>Since <math>y \rightarrow 1</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow 1</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are equal and the two-sided limit exists:</p> $\lim_{x \rightarrow 1} y(x) = 1$ <p>This is identical in concept to the third graph.</p>
	<p>This graph appears to have an “infinite oscillation” on either side of <math>x = 1</math> (the graph continues to bounce between <math>y = 0.5</math> and <math>y = 1.5</math>, regardless of how close you get to <math>x = 1</math>). Since neither one-sided limit approaches a single value, neither exists. Therefore, the two-sided limit is non-existent as well:</p> $\lim_{x \rightarrow 1} y(x) = DNE$
	<p>Since <math>y \rightarrow 1</math> as <math>x \rightarrow 1^-</math> and <math>y \rightarrow 1.5</math> as <math>x \rightarrow 1^+</math>, the one-sided limits are not equal and the two-sided limit does not exist:</p> $\lim_{x \rightarrow 1} y(x) = DNE$ <p>This is another example of a “jump” in the graph. The only difference between this one and the fourth graph is the function value is defined here, although that has nothing to do with the limit.</p>