

The Numerical Derivative

An Introduction

Graphically speaking, a numerical derivative is the slope of a curve at a specific point. This can also be thought of as the slope of the tangent line to the curve at that same point. One common derivative notation is called “prime notation”, where

$$f'(a)$$

means “the numerical derivative of the function $y = f(x)$ at the point $x = a$ ”.

There are four techniques for evaluating numerical derivatives – some result in exact values while others only provide approximations:

- analytically using limit definitions,
- graphically using slopes of tangent lines,
- numerically using the slopes of secant lines, and
- numerically using technology.

Analytically, the numerical derivative of $y = f(x)$ at $x = a$ can be calculated using one of the two limit definitions:

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

It should be noted that there are several rules for shortcutting these analytical limit definitions, but those are beyond the scope of this introduction.

Graphically, a numerical derivative can be approximated by drawing a tangent line to the graph at the point $(a, f(a))$, then calculating the slope using two points on that tangent line.

Given a table of function values, the numerical derivative can be approximated using the closest known points on either side of $x = a$. Simply calculate the slope of the line through those two points (this is essentially the slope of a nearby secant line). If the numerical derivative is needed at an endpoint, use the point $(a, f(a))$ and the next nearest point.

Finally, a calculator can be used to find derivatives numerically. While exact functions vary from calculator to calculator, the command `nDeriv(f(x), x, a)` on the TI-84 will evaluate the numerical derivative of $y = f(x)$ at $x = a$.

Example: Find the numerical derivative of $f(x) = x^2$ at $x = -3$.

Analytically, you can use either version of the limit definition:

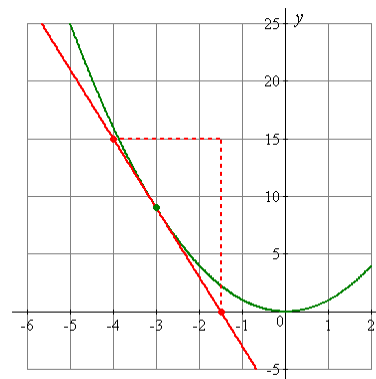
$$\begin{aligned} f'(-3) &= \lim_{\Delta x \rightarrow 0} \frac{f(-3 + \Delta x) - f(-3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(-3 + \Delta x)^2 - (-3)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(9 - 6\Delta x + (\Delta x)^2) - (9)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (-6 + \Delta x) \\ &= -6 + (0) \\ &= -6 \end{aligned}$$

$$\begin{aligned} f'(-3) &= \lim_{x \rightarrow -3} \frac{f(x) - f(-3)}{x - (-3)} \\ &= \lim_{x \rightarrow -3} \frac{x^2 - (-3)^2}{x - (-3)} \\ &= \lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} \\ &= \lim_{x \rightarrow -3} (x - 3) \\ &= (-3) - 3 \\ &= -6 \end{aligned}$$

Graphically, sketch the tangent line to the parabola $y = x^2$ at the point $(-3, 9)$. The line appears to pass through the points $(-4, 15)$ and $(-1.5, 0)$, so the slope can be calculated as

$$f'(-3) = m_{\text{tangent}} = \frac{15 - 0}{-4 - (-1.5)} = -6.$$

Note that this is technically an approximation method, although the result here was the same as the exact value found above with the limit definitions.



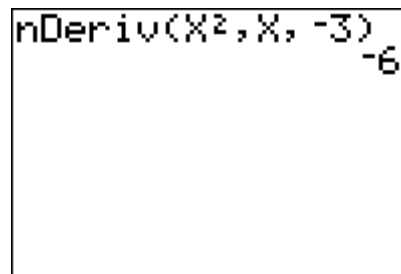
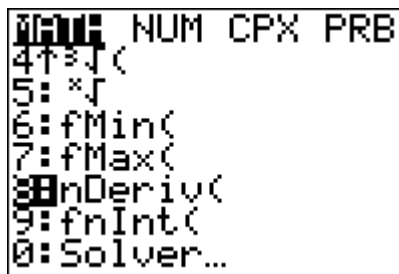
A table of values could be generated for the given function (in practice, this is often the only information available). The numerical derivative could then be approximated by calculating the slope of the secant line that passes through the points on either side of $x = -3$.

x	$f(x)$
-5	25
-4	16
-3	9
-2	4
-1	1

$$f'(-3) \approx \frac{f(-2) - f(-4)}{-2 - (-4)} = \frac{4 - 16}{-2 + 4} = -6$$

It is a coincidence that this approximated value is the same as the exact value (you hope for a reasonable value, but it is rarely exact).

Finally, you can use a graphing utility to evaluate the derivative. On the TI-84, the `nDeriv()` command is found by pressing the MATH key and selecting #8 under the Math menu.



Although the calculator is performing a numerical approximation, it is generally quite accurate.