Multivariable Calculus Portfolio

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Introduction

The following assembly of problems surveys the curriculum of Multivariable Calculus. Included are various subjects ranging from multiple integrals and vector functions all the way to the Divergence Theorem from vector calculus. Each chapter from the book is presented with a short summary of the content and then followed subsequently with a few sample problems that demonstrate the concepts of the chapter's material. Some graphics are provided throughout the portfolio, and they were constructed using Maple 8 and Maple 10 software.

Andrew Kovacs

Chapter 9: Vectors and the Geometry of Space

Introduction

Chapter 9 covered basic vectors, the dot product, the cross product, planar equations, vector functions and their surfaces, cylindrical coordinates, and spherical coordinates.

The first problem demonstrates the use of the dot product in real life. It also touches on placing objects, such as atoms, into 3D space. The application uses these two concepts to create a molecule of methane. I chose this since it demonstrates the use of 3D space and vectors to construct and analyze a wide variety of objects. This kind of construction is also great when working with chemistry since it provides an excellent visual to work with as well as a fundamental grasping of how the atoms fit into space.

The second problem demonstrates an application of spherical geometry as opposed to arbitrary graphing. By using spherical coordinates, the calculation of the distance along the Earth accounts for the curvature. This provides an accurate total displacement, as well as demonstrating various conversions between spherical and rectangular coordinates. It touches briefly on the subject of the cross product, but briefly.

Application

Section 3, problem #37

A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5°.

To begin the problem, make each atom a point in space. A simple configuration is to make the blue Hydrogen atoms at the points (1, 0, 0), (0, 1, 0), (0, 0, 1), and (1, 1, 1). The red Carbon atom centroid then fits in perfectly at (1/2, 1/2, 1/2).

After that, the next step is to make vectors from the Carbon atom to each Hydrogen atom. The grey lines represent these vectors. The orange lines represent the outside shape of the tetrahedron for visualization.

The following image is a graph of the points and vectors made using Maple.

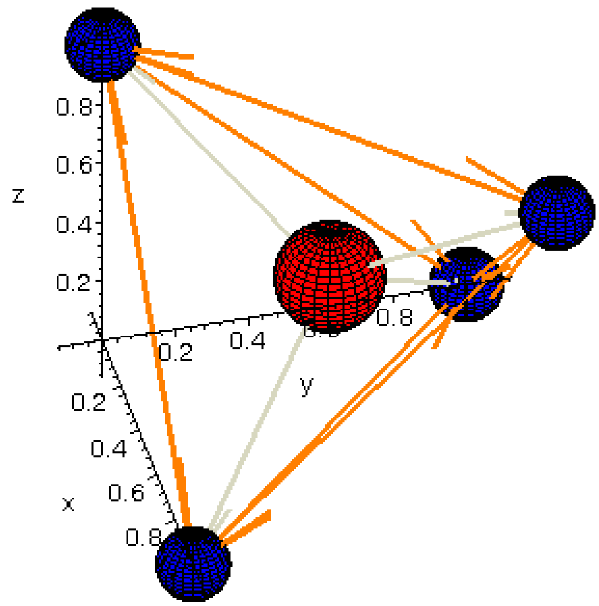

Figure 1: Methane Molecule

The Hydrogen atom points will be designated as follows:

$H_1 = (1, 0, 0)$
$H_2 = (0, 1, 0)$
$H_3 = (0, 0, 1)$
$H_4 = (1, 1, 1)$

Then each vector from the Carbon atom to each Hydrogen atom is:

$$\vec{H}_1 = \left\langle \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2} \right\rangle$$
$$\vec{H}_2 = \left\langle -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle$$
$$\vec{H}_3 = \left\langle -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2} \right\rangle$$
$$\vec{H}_4 = \left\langle \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right\rangle$$

To determine the bond angle a dot product is used with any two vectors. For this problem, H_1 and H_2 were used. To determine the dot product the magnitude of the vectors is needed as well as the actual dot product. Afterwards, algebra will yield the final answer.

$$\vec{H}_1 \bullet \vec{H}_2 = \left\langle \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2} \right\rangle \bullet \left\langle -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2} \right\rangle = \left(\tfrac{1}{2} \cdot -\tfrac{1}{2}\right) + \left(-\tfrac{1}{2} \cdot \tfrac{1}{2}\right) + \left(-\tfrac{1}{2} \cdot -\tfrac{1}{2}\right) = -\tfrac{1}{4}$$

$$\left|\vec{H}_1\right| = \sqrt{(\tfrac{1}{2})^2 + (-\tfrac{1}{2})^2 + (-\tfrac{1}{2})^2} = \sqrt{\tfrac{3}{4}}$$
$$\left|\vec{H}_2\right| = \sqrt{(-\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (-\tfrac{1}{2})^2} = \sqrt{\tfrac{3}{4}}$$

$$\vec{H}_1 \bullet \vec{H}_2 = \left|\vec{H}_1\right| \cdot \left|\vec{H}_1\right| \cdot \cos(\vartheta)$$
$$-\tfrac{1}{4} = \sqrt{\tfrac{3}{4}} \cdot \sqrt{\tfrac{3}{4}} \cdot \cos(\vartheta)$$
$$-\tfrac{1}{4} = \tfrac{3}{4} \cdot \cos(\vartheta)$$
$$-\tfrac{1}{3} = \cos(\vartheta)$$
$$\vartheta = 109.471^\circ \approx 109.5^\circ$$

Application
Section 7, Problem #34

The latitude and longitude of a point *P* in the Northern Hemisphere are related to spherical coordinates ρ, ϑ, ϕ as follows. We take the origin to be the center of the Earth and the positive z-axis to pass through the North Pole. The positive x-axis passes through the point where the prime meridian (the meridian through Greenwich, England) intersects the equator. Then the latitude of *P* is $\alpha = 90^\circ - \phi^\circ$ and the longitude of *P* is $\beta = 360^\circ - \vartheta^\circ$. Find the great-circle distance from Los Angeles (lat. 34.06° N, long. 118.25° W) to Montréal (lat. 45.50° N, long. 73.60° W). Take the radius of the Earth to be 3960 mi. (A *great circle* is the circle of intersection of a sphere and a plane through the center of the sphere).

A simple way to solve this problem is to convert the values given into spherical coordinates and then to convert the results to rectangular coordinates and then find the distance.

To start, we convert the given values into spherical coordinates for Los Angeles:

$$\rho = 3960$$
$$360^\circ - \vartheta = 118.25^\circ$$
$$\vartheta = 241.75^\circ$$
$$90^\circ - \phi = 34.06^\circ$$
$$\phi = 55.94^\circ$$

So the spherical coordinates for Los Angeles are (3960, 241.75°, 55.94°).

Now Montréal:

$$\rho = 3960$$
$$360^\circ - \vartheta = 73.60^\circ$$
$$\vartheta = 286.4^\circ$$
$$90^\circ - \phi = 45.50^\circ$$
$$\phi = 44.50^\circ$$

So the spherical coordinates for Los Angeles are (3960, 286.4°, 44.50°).

Now convert each one into rectangular coordinates, first Los Angeles:

$$x = \rho \sin(\phi)\cos(\vartheta)$$
$$x = 3960\sin(55.94^\circ)\cos(241.75^\circ)$$
$$x = -1552.805$$

$$y = \rho \sin(\phi)\sin(\vartheta)$$
$$y = 3960\sin(55.94^\circ)\sin(241.75^\circ)$$
$$y = -2889.910$$

$$z = \rho \cos(\phi)$$
$$z = 3960\cos(55.94^\circ)$$
$$z = 2217.841$$

So the Los Angeles rectangular coordinates are: (-1552.805, -2889.910, 2217.841).

Next, Montréal:

$$x = \rho \sin(\phi)\cos(\vartheta)$$
$$x = 3960\sin(44.50^\circ)\cos(286.4^\circ)$$
$$x = 783.667$$

$$y = \rho \sin(\phi)\sin(\vartheta)$$
$$y = 3960\sin(44.50^\circ)\sin(286.4^\circ)$$
$$y = -2662.673$$

$$z = \rho \cos(\phi)$$
$$z = 3960\cos(44.50^\circ)$$
$$z = 2824.472$$

So Montréal's rectangular coordinates are (783.667, -2662.673, 2824.472).

Since we have two points, we make each one into a vector from the origin:

$$\vec{L} = < -1552.805, -2889.910, 2217.841 >$$
$$\vec{M} = < 783.667, -2662.673, 2824.472 >$$

The magnitude of each vector is the radius of Earth given from the problem: 3960. Use these figures to calculate the dot product, and then plug the result into the dot product formula and solve for theta:

$$\vec{L} \bullet \vec{M} = (783.667 * -1552.805) + (-2662.673 * -2889.91) + (2824.472 * 2217.841)$$
$$\vec{L} \bullet \vec{M} = 12742233.0984$$

$$\vec{L} \bullet \vec{M} = \left|\vec{L}\right| \cdot \left|\vec{M}\right| \cdot \cos\vartheta$$
$$12742233.0984 = 3960^2 \cos\vartheta$$
$$.8126 = \cos\vartheta$$
$$.6223 = \vartheta$$

Now, apply the arc length formula:

$$s = r \cdot \vartheta$$
$$s = 3960 \cdot .6223$$
$$s = 2464.175$$

So the distance from Los Angeles to Montréal along the curvature of the earth is 2464.175 miles.

Chapter 10: Vector Functions

Introduction

This chapter upgraded the 3D equations with a fundamental tool of calculus: deriving functions and finding related vectors. In particular, the chapter dealt with tangent vectors, normal vectors, binormal vectors, derivatives, normal plane of a curve, and the osculating plane of a curve.

To demonstrate these concepts, I chose a set of equations and applied all of the previous functions to it.

Problem

Using the curve given by $\vec{r}(t) = \langle \sin(2t), t, \cos(2t) \rangle$ find the first and second derivatives, the unit tangent vector, the unit normal vector, the binormal vector, the curvature, the normal plane, and the osculating plane for when $t = \pi/2$:

When the main function is at $t = \pi/2$:

$$\vec{r}\left(\pi/2\right) = \left\langle 0, \pi/2, -1 \right\rangle$$

First and Second Derivative:

$$\vec{r}\,'(t) = \langle 2\cos(2t), 1, -2\sin(2t) \rangle$$
$$\vec{r}\,'\left(\pi/2\right) = \langle -2, 1, 0 \rangle$$
$$\vec{r}\,''(t) = \langle -4\sin(2t), 0, -4\cos(2t) \rangle$$
$$\vec{r}\,''\left(\pi/2\right) = \langle 0, 0, 4 \rangle$$

The derivatives will be used later for the calculation of the various vectors.

The unit tangent vector is the first derivative over the magnitude of the first derivative:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$$\vec{T}(t) = \frac{\langle 2\cos(2t), 1, -2\sin(2t)\rangle}{\sqrt{4\cos^2(2t) + 4\sin^2(2t) + 1}}$$

$$\vec{T}(t) = \frac{\langle 2\cos(2t), 1, -2\sin(2t)\rangle}{\sqrt{5}}$$

$$\vec{T}(t) = \left\langle \frac{2\cos(2t)}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{-2\sin(2t)}{\sqrt{5}} \right\rangle$$

$$\vec{T}\left(\pi/2\right) = \left\langle -\,2/\sqrt{5}, \frac{1}{\sqrt{5}}, 0 \right\rangle$$

The unit normal vector is the derivative of the unit tangent vector divided by its magnitude:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

$$\vec{N}(t) = \frac{\left\langle \frac{-4\sin(2t)}{\sqrt{5}}, 0, \frac{-4\cos(2t)}{\sqrt{5}} \right\rangle}{\sqrt{\frac{16\sin^2(2t)}{5} + \frac{16\cos^2(2t)}{5}}}$$

$$\vec{N}(t) = \frac{\left\langle \frac{-4\sin(2t)}{\sqrt{5}}, 0, \frac{-4\cos(2t)}{\sqrt{5}} \right\rangle}{\sqrt{\frac{16}{5} + \frac{16}{5}}}$$

$$\vec{N}(t) = \frac{\left\langle \frac{-4\sin(2t)}{\sqrt{5}}, 0, \frac{-4\cos(2t)}{\sqrt{5}} \right\rangle}{\sqrt{\frac{32}{5}}}$$

$$\vec{N}(t) = \frac{\left\langle \frac{-4\sin(2t)}{\sqrt{5}}, 0, \frac{-4\cos(2t)}{\sqrt{5}} \right\rangle}{\sqrt{\frac{32}{5}}}$$

$$\vec{N}(t) = \left\langle \frac{-\sin(2t)}{\sqrt{2}}, 0, \frac{-\cos(2t)}{\sqrt{2}} \right\rangle$$

$$\vec{N}\left(\pi/2\right) = \left\langle 0, 0, \frac{1}{\sqrt{2}} \right\rangle$$

The binormal vector is the cross product of the unit tangent and unit normal vectors:

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\vec{B}(t) = \left\langle \frac{2\cos(2t)}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{-2\sin(2t)}{\sqrt{5}} \right\rangle \times \left\langle \frac{-\sin(2t)}{\sqrt{2}}, 0, \frac{-\cos(2t)}{\sqrt{2}} \right\rangle$$

$$\vec{B}(t) = \left\langle -\frac{\cos(2t)}{\sqrt{10}}, \frac{2}{\sqrt{10}}, -\frac{\sin(2t)}{\sqrt{10}} \right\rangle$$

$$\vec{B}\left(\pi/2\right) = \left\langle -\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, 0 \right\rangle$$

Curvature is the magnitude of the derivative of the unit tangent vector over the magnitude of the second derivative of $\vec{r}(t)$:

$$\kappa(t) = \frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$$

$$\kappa(t) = \frac{\frac{4\sqrt{2}}{\sqrt{5}}}{\sqrt{5}}$$

$$\kappa\left(\pi/2\right) = 4\sqrt{2}$$

Normal Plane:

The normal plane contains $\vec{B}(t)$ and $\vec{N}(t)$ and uses $\vec{T}(t)$ as its normal vector.

$$\vec{r}\left(\frac{\pi}{2}\right) = \left\langle 0, \frac{\pi}{2}, -1 \right\rangle$$

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle \frac{-2}{\sqrt{10}}, \frac{2}{\sqrt{10}}, 0 \right\rangle$$

The next step is the plane formula:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In this formula, *a*, *b*, and *c* are the components of the plane's normal vector (in this case, the unit tangent vector), with x_0, y_0, and z_0 representing the initial point. So now plug in the components:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$
$$\frac{-2}{\sqrt{10}}(x-0)+\frac{2}{\sqrt{10}}(y-\frac{\pi}{2})+0=0$$
$$\frac{-2x}{\sqrt{10}}+\frac{2y}{\sqrt{10}}-\frac{\pi}{\sqrt{10}}=0$$
$$\sqrt{10}(-2x+2y-\pi)=0$$

Osculating Plane:

The osculating plane contains of $\vec{T}(t)$ and $\vec{N}(t)$ and uses $\vec{B}(t)$ as its normal vector:

$$\vec{r}\left(\frac{\pi}{2}\right)=\left\langle 0,\frac{\pi}{2},-1\right\rangle$$
$$\vec{B}\left(\frac{\pi}{2}\right)=\left\langle \frac{-1}{\sqrt{10}},\frac{2}{\sqrt{10}},0\right\rangle$$

Again, use the plane formula, but use the binormal vector for *a*, *b*, and *c*:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$
$$\frac{-1}{\sqrt{10}}(x-0)+\frac{2}{\sqrt{10}}(y-\frac{\pi}{2})+0=0$$
$$\frac{-x}{\sqrt{10}}+\frac{2y}{\sqrt{10}}-\frac{\pi}{\sqrt{10}}=0$$
$$\sqrt{10}(-x+2y-\pi)=0$$

This graph represents the given function (red), the first derivative (green), and the second derivative (blue):

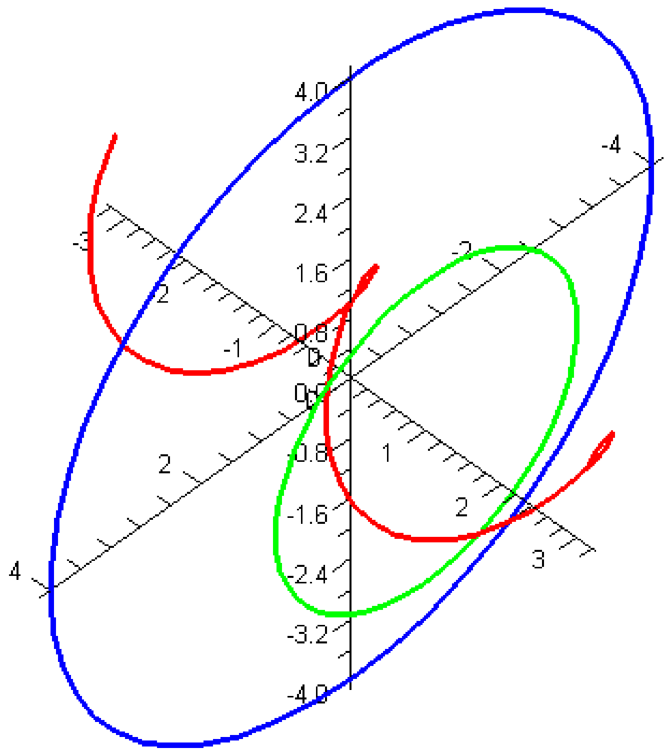

Figure 1: Function and Derivatives

This graph represents the main function and the tangent (red), normal (blue), and binormal (green):

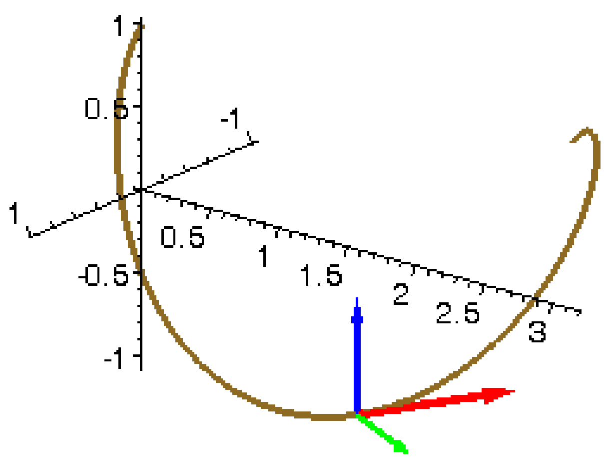

Figure 2: Main Function and Vectors

Chapter 11: Partial Derivatives

Chapter 11 dealt with the more advanced concepts of derivatives and functions of several variables. It included limits, partial derivatives, tangent planes, linear approximations, contour maps, directional derivatives, and the gradient vector. More importantly, the chapter encompassed the chain rule for functions of several variables, how to find the maximum and minimum values in various applications, and Lagrange Multipliers.

Application #1

The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV=8.31T$, where P is measured in kilopascals, V in liters, and T in Kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

This problem is a good example of differentials being used in chemistry. By using the differentials, it is possible to predict the change in any element with the others given. Since we have a volume and temperature, we rewrite the equation as such:

$$P(V,T) = \frac{8.31T}{V}$$

Now find the partials and known values:

$$P_V(V,T) = -\frac{8.31T}{V^2}$$

$$P_T(V,T) = \frac{8.31}{V}$$

$$dV = .3$$

$$dT = -5$$

$$V = 12$$

$$T = 310$$

Using the equation of total differentials, we find:

$$dP = P_V dV + P_T dT$$

$$dP = -\frac{8.31T}{V^2} \cdot .3 + \frac{8.31}{V} \cdot -5$$

Now solve the equation for the change in pressure:

$$dP = -\frac{8.31T}{V^2}\cdot .3 + \frac{8.31}{V}\cdot -5$$

$$dP = -\frac{8.31\cdot 310}{12^2}\cdot .3 + \frac{8.31}{12}\cdot -5$$

$$dP = -5.367 + -3.46$$

$$dP = -8.827$$

So the approximate change in pressure would be -8.827 kilopascals, or it would drop by 8.827 kilopascals.

Application #2

A right circular cone is being filled with water. The radius is increasing at a rate of 1.8 inches per second and the height is increasing by 2.5 inches per second. At what rate is the volume of the cone changing when the radius is 120 inches and the height is 140 inches?

This problem deals with related rates. However, previous calculus classes teach this subject with only one variable. By using the chain rule and differentials, it is possible to obtain the answer using all of the variables.

First, rewrite to equation for the volume of a right circular cone as a function of several variables and find the partial derivatives:

$$V(r,h) = \tfrac{1}{3}\pi\cdot r^2\cdot h$$

$$V_r(r,h) = \tfrac{2}{3}\pi\cdot r\cdot h$$

$$V_h(r,h) = \tfrac{1}{3}\pi\cdot r^2$$

The rate of change for the radius and height are given:

$$\frac{dr}{dt} = 1.8$$

$$\frac{dh}{dt} = 2.5$$

Now we can use the chain rule:

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$\frac{dV}{dt} = \tfrac{2}{3}\pi\cdot r\cdot h\cdot 1.8 + \tfrac{1}{3}\pi\cdot r^2\cdot 2.5$$

Now, plug in the given radius and height and solve:

$$\frac{dV}{dt} = \tfrac{2}{3}\pi \cdot r \cdot h \cdot 1.8 + \tfrac{1}{3}\pi \cdot r^2 \cdot 2.5$$
$$\frac{dV}{dt} = \tfrac{2}{3}\pi \cdot 120 \cdot 140 \cdot 1.8 + \tfrac{1}{3}\pi \cdot 120^2 \cdot 2.5$$
$$\frac{dV}{dt} = 63{,}334.508 + 37{,}699.112$$
$$\frac{dV}{dt} = 101{,}033.62$$

So the rate of change of the volume is 101,033.62 cubed inches per second. Converting this to gallons yields the rate of 437.37 gallons per second.

Application #3

Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.

To solve this problem, Lagrange Multipliers are needed. To use Lagrange Multipliers, let's set up a system of equations for our use. We can use the volume equation and the constraints:

$$f(x,y,z) = x \cdot y \cdot z$$
$$4x + 4y + 4z = 200$$
$$2x \cdot y + 2x \cdot z + 2y \cdot z = 1500$$

The second equation will be g(x, y, z) and the third h(x, y, z). These will simplify too:

$$g(x,y,z) = x + y + z = 50$$
$$h(x,y,z) = x \cdot y + x \cdot z + y \cdot z = 750$$

And now let's get ∇f, ∇g, and ∇h:

$$\nabla f(x,y,z) = \langle y \cdot z, x \cdot z, x \cdot y \rangle$$
$$\nabla g(x,y,z) = \langle 1,1,1 \rangle$$
$$\nabla h(x,y,z) = \langle y + z, x + z, x + y \rangle$$

Lagrange Multipliers are a special kind of system of equations that can be used to maximize and minimize multivariable functions. Now for Lagrange Multipliers:

$$\nabla f(x,y,z) = \lambda \cdot g(x,y,z) + \mu \cdot h(x,y,z)$$
$$\nabla f_x = \lambda \cdot g_x + \mu \cdot h_x$$
$$\nabla f_y = \lambda \cdot g_y + \mu \cdot h_y$$
$$\nabla f_z = \lambda \cdot g_z + \mu \cdot h_z$$
$$g(x,y,z) = k$$
$$h(x,y,z) = l$$

And here they are with the variables filled in:

$$y \cdot z = \lambda \cdot y + \lambda \cdot z + \mu$$
$$x \cdot z = \lambda \cdot x + \lambda \cdot z + \mu$$
$$x \cdot y = \lambda \cdot x + \lambda \cdot y + \mu$$

$$x + y + z = 50$$
$$x \cdot y + x \cdot z + y \cdot z = 750$$

To solve the system just multiply everything by the missing variable and set a pair equal to each other:

$$x \cdot y \cdot z = \lambda \cdot x \cdot y + \lambda \cdot x \cdot z + \mu$$
$$x \cdot y \cdot z = \lambda \cdot x \cdot y + \lambda \cdot y \cdot z + \mu$$
$$x \cdot y \cdot z = \lambda \cdot x \cdot z + \lambda \cdot y \cdot z + \mu$$

Now solve the system:

$$\lambda \cdot x \cdot y + \lambda \cdot x \cdot z + \mu = \lambda \cdot x \cdot y + \lambda \cdot y \cdot z + \mu$$
$$\lambda \cdot x \cdot z = \lambda \cdot y \cdot z$$
$$x = y$$

$$\lambda \cdot x \cdot y + \lambda \cdot y \cdot z + \mu = \lambda \cdot x \cdot z + \lambda \cdot y \cdot z + \mu$$
$$\lambda \cdot x \cdot y = \lambda \cdot x \cdot z$$
$$y = z$$

$$x + y + z = 50$$
$$3 \cdot x = 50$$
$$x = \frac{50}{3}$$

So now we know that all the sides equal $^{50}\!/_{3}$. However, if we plug this into the other constraints it fails:

$$x \cdot y + x \cdot z + y \cdot z = 750$$
$$\frac{50}{3} \cdot \frac{50}{3} + \frac{50}{3} \cdot \frac{50}{3} + \frac{50}{3} \cdot \frac{50}{3} = \frac{2500}{3} \neq 750$$

So the box is not a cube. Instead assume that one of the sides, for now *x*, is different from the others. At this point you then subtract two of the original multipliers from each other:

$$y \cdot z - x \cdot z = \lambda \cdot y + \lambda \cdot z + \mu - \lambda \cdot x - \lambda \cdot z - \mu$$
$$z \cdot (y - x) = \lambda \cdot y - \lambda \cdot x$$
$$z \cdot (y - x) = \lambda \cdot (y - x)$$
$$z = \lambda$$

Now use another two:

$$y \cdot z - x \cdot y = \lambda \cdot y + \lambda \cdot z + \mu - \lambda \cdot x - \lambda \cdot y - \mu$$
$$y \cdot (z - x) = \lambda \cdot z - \lambda \cdot x$$
$$y \cdot (z - x) = \lambda \cdot (z - x)$$
$$y = \lambda$$

Now that *y* and *z* are equal, plug them into the constraints:

$$x + y + z = 50$$
$$x + 2y = 50$$

$$x \cdot y + x \cdot z + y \cdot z = 750$$
$$2x \cdot y + y^2 = 750$$

Solve these constraints for *x*:

$$x = 50 - 2y$$
$$x = \frac{750 - y^2}{2y}$$

Now set them equal to each other and simplify:

$$\frac{750-y^2}{2y}=50-2y$$
$$750-y^2=100-4y^2$$
$$\frac{750-y^2}{2y}=50-2y$$

Upon using the quadratic formula...

$$y=\frac{-b\pm\sqrt{b^2-4a\cdot c}}{2a}$$
$$y=\frac{100\pm\sqrt{100^2-4\cdot 3\cdot 750}}{6}$$
$$y=\frac{100\pm\sqrt{10000-9000}}{6}$$
$$y=\frac{100\pm\sqrt{1000}}{6}$$
$$y=\frac{100\pm 10\sqrt{10}}{6}$$
$$y=\frac{50\pm 5\sqrt{10}}{3}$$

So the maximum and minimum values of the box are at these points:

$$\left(\frac{50+5\sqrt{10}}{3},\frac{50-5\sqrt{10}}{3},\frac{50-5\sqrt{10}}{3}\right)$$
$$\left(\frac{50-5\sqrt{10}}{3},\frac{50+5\sqrt{10}}{3},\frac{50+5\sqrt{10}}{3}\right)$$

So the minimum is:

$$f\left(\frac{50+5\sqrt{10}}{3},\frac{50-5\sqrt{10}}{3},\frac{50-5\sqrt{10}}{3}\right)=\frac{87500-2500\sqrt{10}}{27}$$

And the maximum is:

$$f\left(\frac{50-5\sqrt{10}}{3},\frac{50+5\sqrt{10}}{3},\frac{50+5\sqrt{10}}{3}\right)=\frac{87500+2500\sqrt{10}}{27}$$

So the minimum volume of the box is 2947.94 cm^3 and the maximum is 3533.54 cm^3. In solving the problem, we used x as our different variable. However, it should be noted that any variable could be assumed to be different. In doing so the only difference is that we would have had to pick a different set of equations to subtract from each other.

Chapter 12: Multiple Integrals

This chapter dealt with using double and triple integrals to expand integral calculus into multiple dimensions. Double integrals will calculate the volume over a 2D region, and triple integrals can calculate an abstract 4th dimension from a 3D region. The chapter also covered double and triple integrals over general regions, surface area, and converting multivariable integrals written in polar, cylindrical, and spherical form.

Problem #1

Evaluate $\iint_D x \cdot y \cdot dA$ for the following region:

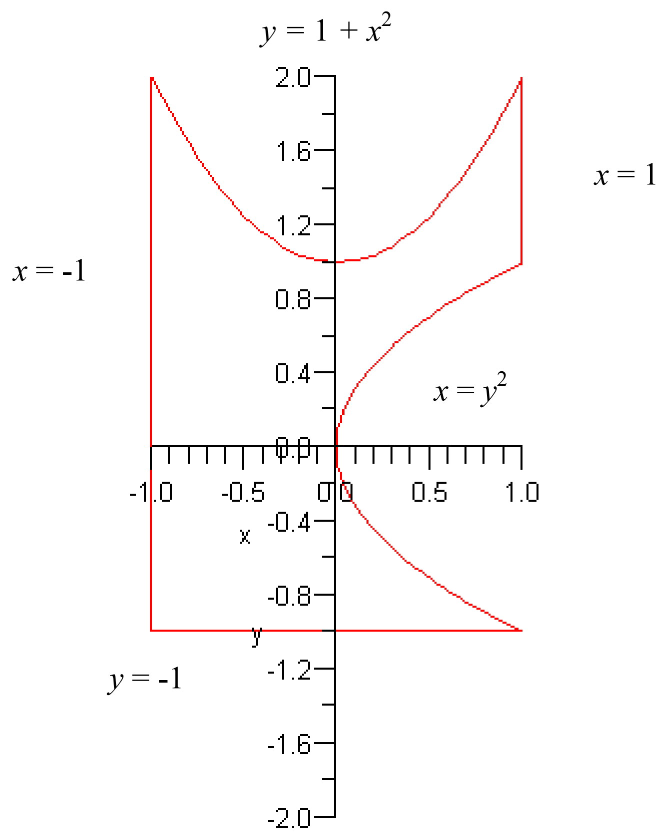

Figure 1: Integral Region

This problem demonstrates how to set up and solve multiple integrals and can be done by separating the double integral into multiple sections. Let's try doing one integral for each quadrant.

Quadrant #1:

Let's set up our rectangles to be vertical. This way, the upper bound will be $y = 1 + x^2$ and the lower bound $y = \sqrt{x}$.

$$A_1 = \int_0^1 \int_{\sqrt{x}}^{1+x^2} x \cdot y dy dx$$

$$A_1 = \int_0^1 \left[\frac{x \cdot y^2}{2} \right]_{\sqrt{x}}^{1+x^2} dx$$

$$A_1 = \int_0^1 \left(\frac{x \cdot (1+x^2)^2}{2} - \frac{x^2}{2} \right) dx = \int_0^1 \left(\frac{x^5 + 2x^3 + x}{2} - \frac{x^2}{2} \right) dx$$

$$A_1 = \left[\frac{x^6}{12} + \frac{x^4}{4} + \frac{x^2}{4} - \frac{x^3}{6} \right]_0^1 = \frac{10}{24} = \frac{5}{12}$$

Quadrant #2:

If we make the rectangles vertical again, the upper bound can be $y = 1 + x^2$ and the lower bound $y = 0$.

$$A_2 = \int_{-1}^0 \int_0^{1+x^2} x \cdot y dy dx$$

$$A_2 = \int_{-1}^0 \left[\frac{x \cdot y^2}{2} \right]_0^{1+x^2} dx$$

$$A_2 = \int_{-1}^0 \left(\frac{x \cdot (1+x^2)^2}{2} - 0 \right) dx$$

$$A_2 = \left[\frac{x^6}{12} + \frac{x^4}{4} + \frac{x^2}{4} \right]_{-1}^0 = -\frac{7}{12}$$

Quadrant #3:

This fragment is just a square.

$$A_3 = \int_{-1}^{0}\int_{-1}^{0} x \cdot y dy dx$$

$$A_3 = \int_{-1}^{0}\left[\frac{x \cdot y^2}{2}\right]_{-1}^{0} dx$$

$$A_3 = \left[-\frac{x^2}{4}\right]_{-1}^{0} = \frac{1}{4}$$

Quadrant #4:

Once more, vertical rectangles with an upper bound of $y = -\sqrt{x}$ and a lower bound of $y = -1$.

$$A_4 = \int_{0}^{1}\int_{-1}^{-\sqrt{x}} x \cdot y dy dx$$

$$A_4 = \int_{0}^{1}\left[\frac{x \cdot y^2}{2}\right]_{-1}^{-\sqrt{x}} dx$$

$$A_4 = \left[\frac{x^2}{2} - \frac{x}{2}\right]_{-1}^{-\sqrt{x}} = -\frac{1}{12}$$

So the total is:

$$T = \frac{5}{12} - \frac{7}{12} + \frac{1}{4} - \frac{1}{12}$$

$$T = \frac{5}{12} + \frac{3}{12} - \frac{8}{12}$$

$$T = 0$$

Since we are evaluating an integral, the total can be zero.

Problem #2

Evaluate the integral:

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dzdxdy$$

This integral, if solved in rectangular coordinates, would be difficult (not to mention unsightly). However, by converting to spherical coordinates we can simplify it. First, use the bounds given to find out the bounds for ρ:

$$z = \sqrt{18-x^2-y^2}$$
$$z^2 = 18-x^2-y^2$$
$$x^2+y^2+z^2 = 18$$
$$\rho^2 = 18$$
$$\rho = \sqrt{18}$$

Next, convert the main equation:

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dzdxdy$$
$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{18}} (\rho^2)\cdot\rho^2\cdot\sin(\phi)\,d\rho\cdot d\phi\cdot d\theta$$

And then solve:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{18}} (\rho^2)\cdot\rho^2\cdot\sin(\phi)\,d\rho\cdot d\phi\cdot d\theta$$
$$2\pi\cdot\int_0^{\pi} \int_0^{\sqrt{18}} (\rho^2)\cdot\rho^2\cdot\sin(\phi)\,d\rho\cdot d\phi$$
$$2\pi\cdot\int_0^{\pi}\left[\frac{\rho^5}{5}\sin(\phi)\right]_0^{\sqrt{18}} d\phi$$
$$2\pi\cdot\frac{\sqrt{18}^5}{5}\cdot\int_0^{\pi}[\sin(\phi)]d\phi$$
$$2\pi\cdot\frac{\sqrt{18}^5}{5}\cdot[-\cos(\phi)]_0^{\pi}$$
$$2\pi\cdot\frac{\sqrt{18}^5}{5}\cdot 2$$
$$4\pi\cdot 324\cdot\frac{\sqrt{18}}{5}$$

Chapter 13: Vector Calculus

Chapter 13 finished the textbook used in our class and dealt with one of the arguably more difficult fields of calculus: Vector Calculus. After discussing vector fields, the chapter introduces line integrals and surface integrals. The chapter also connects different integrals in with Green's Theorem, Stoke's Theorem, and the Divergence Theorem.

Problem #1

Find the work done by the force field $\vec{F}(x,y)=(x^2)\vec{i}+(x\cdot y)\vec{j}$ on a particle that moves once around the circle $x^2+y^2=4$ oriented in the counterclockwise direction.

This problem is a question of work, and as such can be solved with a line integral:

$$\int_c \vec{F}\bullet d\vec{r}=\int_c \vec{F}(\vec{r}(t))\bullet\vec{r}\,'(t)dt$$

To use the line integral in vector format, there needs to be the original field (provided by $\vec{F}(x,y)$), a path along curve C defined as $\vec{r}(t)$, and the first derivative of the path, $\vec{r}(t)$. After we have all three, we just have to plug them into the formula. Since the path is a circle with radius two:

$$\vec{F}(x,y)=\langle x^2, x\cdot y\rangle$$
$$\vec{r}(t)=\langle 2\cos(t), 2\sin(t)\rangle$$
$$\vec{r}\,'(t)=\langle -2\sin(t), 2\cos(t)\rangle$$

The bounds for t are:

$$0\le t\le 2\pi$$

Next, plug them into the line integral formula:

$$\int_c \vec{F}(\vec{r}(t))\bullet\vec{r}\,'(t)dt=\int_0^{2\pi}\langle 4\cos^2(t), 4\cos(t)\sin(t)\rangle\bullet\langle -2\sin(t), 2\cos(t)\rangle dt$$

Now, solve the integral:

$$\int_0^{2\pi} \langle 4\cos^2(t), 4\cos(t)\sin(t) \rangle \bullet \langle -2\sin(t), 2\cos(t) \rangle dt$$

$$\int_0^{2\pi} \left(8\cos^2(t)\sin(t) - 8\cos^2(t)\sin(t)\right) dt$$

$$\int_0^{2\pi} 0 dt = 0$$

So the answer is zero, which means that the field performs no work on the particle. To explain this, look the graph of the field and path from above. In this graph, the vector field is represented by green arrows and the path *C* that the particle moves along is represented by the dark blue circle:

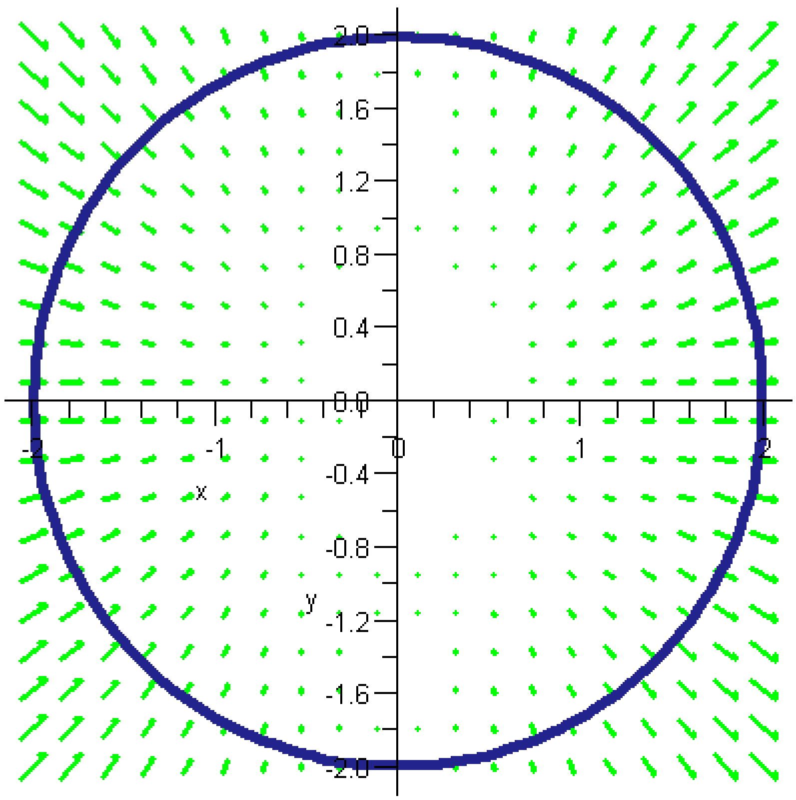

Figure 1: Vector Field and Path *C*

By looking at this graph, we can tell that the vector field is always perpendicular to the path *C* in opposing directions thus canceling the work over the entire path. This is why the line integral, which calculates work, gives a result of zero. However, even though the integral yields zero the field is not conservative.

Problem #2

If $\vec{F}(x,y,z)=\langle 2z,4x,5y\rangle$, evaluate $\int_c \vec{F}\bullet d\vec{r}$ along curve *C*. Curve *C* is a curve represented by the intersection of the plane $z=x+4$ with the cylinder $x^2+y^2=4$.

Setting up a line integral to follow that curve would take a considerable amount of time, and involve far more effort than is needed to solve the problem. However, Stoke's Theorem lets us take a minimalist approach and still produce the answer. Stoke's Theorem states:

$$\int_c \vec{F}\bullet d\vec{r}=\iint_S curl(\vec{F})\bullet d\vec{S}=\iint_S curl(\vec{F})\bullet(\vec{r}_u x\vec{r}_v)dA$$

So instead of drawing out a long complicated line integral, we can instead use the curl. Let's calculate $(\vec{r}_u\times\vec{r}_v)$:

$$\vec{r}(r,\theta)=\langle r\cos(\theta),r\sin(\theta),r\cos(\theta)+4\rangle$$
$$\vec{r}_r=\langle\cos(\theta),\sin(\theta),\cos(\theta)\rangle$$
$$\vec{r}_\theta=\langle -r\sin(\theta),r\cos(\theta),-r\sin(\theta)+4\rangle$$
$$\vec{r}_r\times\vec{r}_\theta=\langle r\cos^2(\theta)+r\sin^2(\theta),r\cos(\theta)\sin(\theta)-r\cos(\theta)\sin(\theta),r\cos^2(\theta)+r\sin^2(\theta)\rangle$$
$$\vec{r}_r\times\vec{r}_\theta=\langle r,0,r\rangle$$

Our bounds will be $0\le r\le 2$ and $0\le\theta\le 2\pi$. Now calculate $curl(\vec{F})$:

$$curl(\vec{F})=\left\langle \partial/\partial x,\partial/\partial y,\partial/\partial z\right\rangle x\langle 2z,4x,5y\rangle$$
$$curl(\vec{F})=\langle 5,2,4\rangle$$

Now plug $curl(\vec{F})$ and $(\vec{r}_u x\vec{r}_v)$ into the equation:

$$\iint_S curl(\vec{F})\bullet(\vec{r}_u x\vec{r}_v)dA=\int_0^{2\pi}\int_0^2\langle 5,2,4\rangle\bullet\langle r,0,r\rangle drd\theta$$

Now solve:

$$\int_0^{2\pi}\int_0^{2}\langle 5,2,4\rangle \bullet \langle r,0,r\rangle dr d\theta$$

$$\int_0^{2\pi}\int_0^{2}(5r+4r)dr d\theta$$

$$2\pi\int_0^{2}(9r)dr = 36\pi$$

So our answer is 36π.

Problem #3

If $\vec{F}(x,y,z) = \langle 3x\cdot y^2, x\cdot e^z, z^3\rangle$, calculate $\iint_S (\vec{F})\bullet d\vec{S}$, or the flux of $\vec{F}$ across the surface *S*. *S* is the surface of the solid bound by the cylinder $y^2+z^2=1$ and the planes $x=-1$ and $x=2$.

Rather than calculating the flux with nothing short of convoluted algebra and calculus, we can simplify this problem using Divergence Theorem which states:

$$\iint_S (\vec{F})\bullet d\vec{S} = \iiint_E div(\vec{F})dV$$

In this equation *E* is the solid bounded by the closed surface *S*. To set up the Divergence Theorem, we only need to calculate $div(\vec{F})$:

$$div(\vec{F}) = \left\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \right\rangle \bullet \langle 3x\cdot y^2, x\cdot e^z, z^3\rangle$$

$$div(\vec{F}) = 3y^2 + 3z^2$$

Now, plug it in:

$$\iiint_E div(\vec{F})dV = \iiint_E 3y^2 + 3z^2 dV$$

We know that $y^2+z^2=1$, so we can simplify our current equation even further and convert it to cylindrical:

$$\iiint_E 3dV = \int_{-1}^{2}\int_0^{2\pi}\int_0^{1} 3r\cdot dr\cdot d\theta\cdot dz = 9\pi$$

And the answer is 9π.

Tests Section

The following attached papers are the several tests taken during the class.