A Graphical Approach for Finding Points of Inflection

Introduction

In precalculus, we have defined a point of inflection as the coordinate on a graph where concavity changes from upward to downward or from downward to upward. Identifying the concavity in a graph is relatively easy, but locating exactly where it changes can be quite difficult!

In calculus, a concept called the derivative can be used to find the exact location of all points of inflection in any function. Although derivatives are not presented in precalculus, we can use its idea with the graphing calculator to at least approximate points of inflection to several decimal places.

A Little Background

Recall that upward concavity can be described as when the slope of a curve is increasing (that is, the slope either gets more positive or less negative as x increases). Similarly, downward concavity is when the slope of a curve is decreasing (the slope gets less positive or more negative).

Since a point of inflection is where there is a change in the concavity of a curve, we could also consider that a point of inflection is where the slope of that curve changes from increasing to decreasing or decreasing to increasing.

Also recall that a relative maximum will occur when a curve changes from increasing to decreasing. A relative minimum will occur when a curve changes from decreasing to increasing.

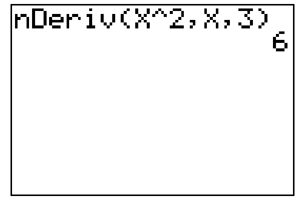
Putting this all together, if we can create a graph of the *slope* of the original curve (which is also known as the *derivative* of the original curve), we could find its relative extrema on the calculator. This would tell us where the slope of the original curve changes from increasing to decreasing (or decreasing to increasing), which means the concavity of the original curve changes from upward to downward (or downward to upward). The location (in terms of x only) of these relative extrema would be identical to the location of the points of inflection in the original graph.

Of course, we cannot easily generate a graph of the slope of the original curve by hand. It turns out the graphing calculator can, so we can let it do all of the work!

The nDeriv Command

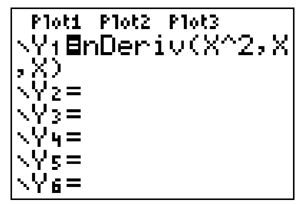
On the TI-83/84, there is a command called nDeriv which can be located by pressing the MATH button and selecting 8:nDeriv(. This command will approximate the slope of a curve at any point and requires three parameters: the function defining the curve, the variable being used for the domain, and the point in the domain where the slope is desired.

For example, suppose you want the slope of the curve $y = x^2$ at the coordinate (3, 9). You would need to enter the function (X^2), the variable (X), and the location (3) as shown at the right. Press ENTER and the calculator estimates the slope as 6 (although in this case the approximation is exact).

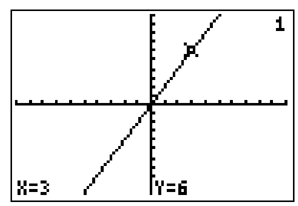


Of course, we do not want the slope of a curve at a single point, but rather a graph of all slopes. What we have to do to accomplish this is to use the nDeriv command in the graphing area of the calculator, treating it like its own function in terms of *x*. The secret is to replace the last parameter with “X”. This essentially tells the calculator to plug in every *x*-value it can, calculate that slope, and then graph the result as (*x*, *y*) where *y* is that slope value for each *x*.

Let’s revisit $y = x^2$. We know its slope at $x = 3$ is 6, but what is its slope everywhere? Press Y= and enter the nDeriv command as shown. Next, press GRAPH and you will get a curve (in this case, a line) that is a “collection” of all the slopes in the viewing window.

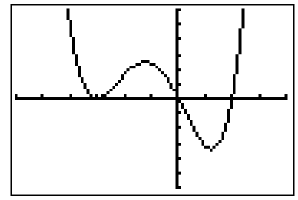


You can now use this function just as you would any other in the calculator. For example, press TRACE and you can check its value at $x = 3$. Notice the *y*-value is 6. This means the slope of the original curve at $x = 3$ is 6, matching our result from the first computation on the Home screen.



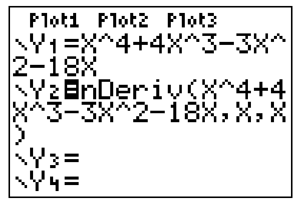
In this example, you should notice how the slope graph (the derivative) is a line that is always increasing. This means the original curve always has upward concavity and, therefore, no point of inflection. Of course, we already knew that since the original function was the parent quadratic function and we are familiar with its parabolic behavior.

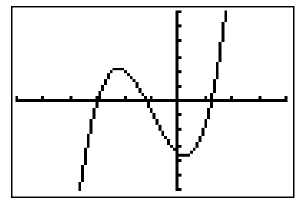
Locating Points of Inflection



Okay, we can finally get to the task at hand: finding points of inflection. Let's consider the polynomial function $f(x) = x^4 + 4x^3 - 3x^2 - 18x$. For starters, let's create a graph of this function to get an idea of its concavity. The one shown has a viewing window of $[-6, 4] \times [-30, 30]$.

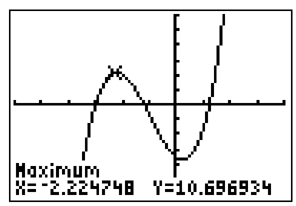
From this graph, we can see its concavity is initially upward, then downward, and then upward again. This means that it has two points of inflection, but they can only be approximated at this point. A guess might be at the points $(-2, 8)$ and $(0, 0)$.

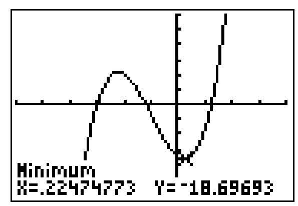




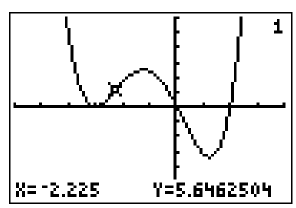
Now let's use the nDeriv command to create a graph of the slopes of $f(x)$. Enter the command in Y2 as shown (notice Y1 is turned off), then press the GRAPH button. The result is a graph of the slopes of $f(x)$, formally referred to as the derivative of $f(x)$.

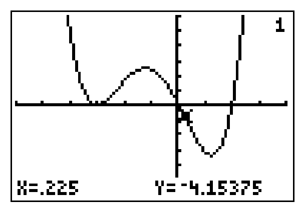
Notice how the slope graph is initially increasing, then decreasing, and then increasing again. This correlates to the concavity behavior in the graph of $f(x)$. Also notice the two relative extrema, one maximum and one minimum. Use the calculator's tools to locate the two extrema (you really only need the x-value).





According to this graph, the two extrema occur at the values $x = -2.225$ and $x = 0.225$. This is also where the points of inflection will occur in the graph of $f(x)$.





To find the actual coordinates for the points of inflection, you can either substitute these x-values into the function or graph the original again at trace the points. Doing so shows the points of inflection are $(-2.225, 5.646)$ and $(0.225, -4.154)$.

You can see our initial approximation was close, but the calculator's is better. To be clear, even the calculator's answers are estimations. It would take calculus to find the exact answers, which happen to be $\left(-1-\frac{\sqrt{6}}{2}, \frac{3}{4}+2\sqrt{6}\right)$ and $\left(-1+\frac{\sqrt{6}}{2}, \frac{3}{4}-2\sqrt{6}\right)$. Yikes!