

A Summary of Graphical Analysis

The details below, while abbreviated, should help the typical algebra student in analyzing the graphs of common functions. It focuses primarily on algebraic functions (those involving polynomials, radicals, and rationals). Some of the generalizations would not apply for transcendentals (exponentials, logarithms, and trigonometrics) or piecewise functions.

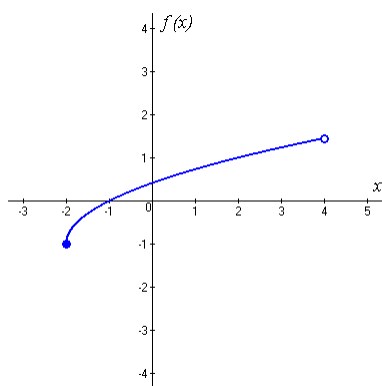
Domain

The domain is the set of x -values that can be used in the function to get an output. The domain is often stated as an interval of these x -values, although set builder notation is used as well.

Graphically, the domain can be found moving along the x -axis (from left to right) and looking up or down. If the curve is above or below that point on the x -axis, that x -value is part of the domain.

Example

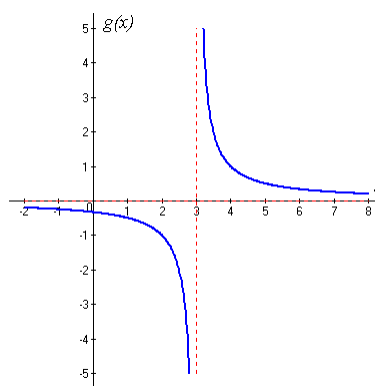
Consider the functions $f(x)$ and $g(x)$ graphed below.



Domain: $[-2, 4)$

or

$\{x \mid -2 \leq x < 4\}$



Domain: $(-\infty, 3) \cup (3, \infty)$

or

$\{x \mid x \neq 3\}$

Since the graph of $f(x)$ begins with $x = -2$ and is continuous until $x = 4$, its domain would include all values between -2 and 4 . It would also include $x = -2$ since there is a closed circle at that endpoint; the domain would not include $x = 4$ since there is an open circle.

The graph of $g(x)$ extends forever to the left and to the right, so its domain would be unbounded (approaches infinity in both directions). Because the function is not defined at $x = 3$ (there is a vertical asymptote in the graph), this value must be excluded from the domain.

Domain (continued)

Algebraically, it is often easier to start with the assumption the domain will be all real numbers and then exclude values that will not work in the function rule. Specifically, look for functions that have denominators or square roots. Exclude x -values that make denominators zero or radicands negative from the domain.

It is worth noting there are non-algebraic functions without denominators or radicals that exclude values as well. For example, logarithmic functions must exclude values that make its argument non-positive. Also, there are some trigonometric functions that are undefined at specific (and periodically repeated) values.

Example

Suppose the function $f(x) = \frac{\sqrt{x+6}}{x-2}$, which contains both a radical and a denominator. Start by considering the fact that the radicand, $x + 6$, must be non-negative.

$$\begin{aligned}x + 6 &\geq 0 \\x &\geq -6\end{aligned}$$

This means that all values less than -6 must be excluded from the domain of the function $f(x)$. Now consider the fact that the denominator, $x - 2$, cannot be zero.

$$\begin{aligned}x - 2 &\neq 0 \\x &\neq 2\end{aligned}$$

From this, $x = 2$ must also be excluded from the domain. Combining that facts that $x \geq -6$ and $x \neq 2$, the domain of the function $f(x)$ can be described using interval or set builder notation.

$$\begin{aligned}\text{Domain: } &[-6, 2) \cup (2, \infty) \\ &\text{or} \\ &\{x \mid x \geq -6 \text{ and } x \neq 2\}\end{aligned}$$

Range

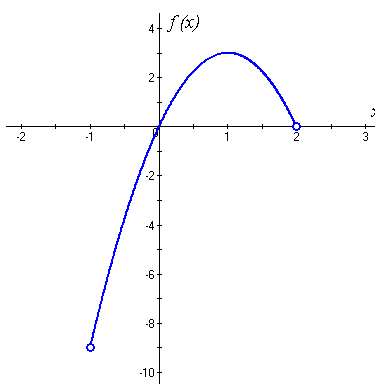
The range is the set of y -values that can be outputs for the function. Like domain, the range can be stated in interval or set builder notation.

Graphically, the range can be found moving along the y -axis (from down to up) and looking left or right. If the curve is to either side of that point on the y -axis, that y -value is part of the range. Notice this is the same process as with domain, only your moving vertically and answering with y -values.

Generally speaking, there is not a succinct algebraic technique for finding the range of a function. A combination of facts including the transformation of basic graph shapes (not addressed here) and also the increasing, decreasing, and end behaviors on the domain all play into describing the range of a function given its algebraic rule.

Example

Consider the function $f(x)$ graphed below.



Range: $(-9, 3]$

or

$\{y \mid -9 < y \leq 3\}$

Since the graph of $f(x)$ begins with $y = -9$ and is continuous until $y = 3$, its range would include all values between -9 and 3 . It would not include $y = -9$ since there is an open circle there, but the range would include $y = 3$ since the curve touches the point $(1, 3)$.

Notice that the endpoint at $(2, 0)$ is open, thus not part of the function. Since the point $(0, 0)$ is on the graph, however, $y = 0$ is still included in the range.

Intercepts

An x -intercept is a point where the function's graph touches the x -axis. There can be multiple x -intercepts, and it is best to identify them as coordinates in the form $(a, 0)$. When using a graphing utility, an x -intercept can usually be approximated using a "zero" function.

Since an x -intercept corresponds to the zero or root of the function being graphed, the x -coordinate of the intercept can be found algebraically by setting the function equal to zero and solving for x . Solutions can require a variety of algebraic techniques, but factoring is typical.

The y -intercept is where the function's graph touches the y -axis. A function cannot have more than one y -intercept, and it is best to identify it as a coordinate in the form $(0, b)$. When using a graphing utility, a y -intercept can usually be approximated using the "trace" feature.

Algebraically, the y -coordinate of the y -intercept can be found by evaluating the function for $x = 0$.

Example

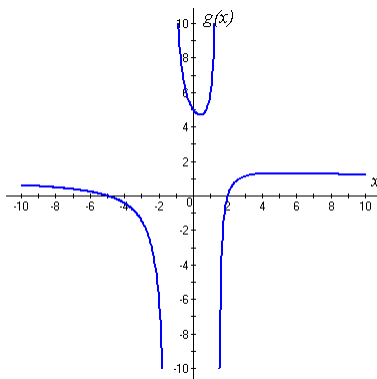
Consider the function $g(x) = \frac{x^2 + 3x - 10}{x^2 - 2}$. Its x -intercepts can be found by locating the zeros of the function, which will occur when the rational expression's numerator is equal to zero.

$$\begin{aligned}x^2 + 3x - 10 &= 0 \\(x + 5)(x - 2) &= 0 \\x &= -5, 2\end{aligned}$$

Since the function has zeros at $x = -5$ and $x = 2$, there are x -intercepts at $(-5, 0)$ and $(2, 0)$.

For the y -intercept, evaluate the function when $x = 0$. Since $g(0) = 5$, there is a y -intercept at $(0, 5)$.

All three of these intercepts can be confirmed graphically as well.



Vertical Asymptotes and Holes

A vertical asymptote essentially describes where a function is unbounded, or gets infinitely large in either the positive or negative direction. Graphically, a vertical asymptote looks like an imaginary vertical line that the function never crosses, but instead gets very close to while going up or down in the y -direction forever. A vertical asymptote is identified as a linear equation, $x = a$.

A single point on the graph of a function where the curve is undefined is called a hole. Unlike a vertical asymptote, a hole is infinitely small and cannot be seen on a graph, so it is identified with an open circle when being drawn. Holes are usually identified with coordinates, but it is important to note that the coordinates do not represent a point on the graph.

In calculus, limits are used to define vertical asymptotes and holes for any function. In algebra, however, rational functions will have a vertical asymptote when a value for x causes the denominator to be zero and the numerator to be non-zero.

If both the denominator and numerator are zero – which will occur when the top and bottom of the rational function have identical factors – then there will be a hole in the graph. The coordinate for the hole can actually be located by reducing those common factors and evaluating the new expression for the same x -value.

Example

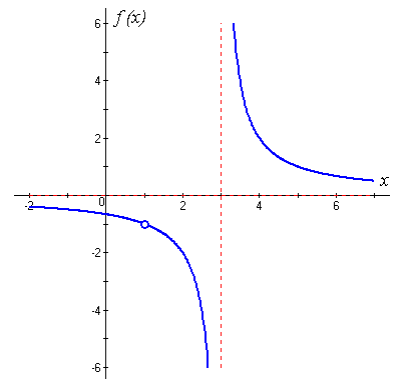
Consider the function $f(x) = \frac{2x-2}{x^2-4x+3}$, which can be factored as $f(x) = \frac{2(x-1)}{(x-1)(x-3)}$.

Notice $x = 1$ would cause the numerator and denominator to both be zero, so there must be a hole. To find its coordinate, reduce the expression and substitute $x = 1$. Doing so shows there is a hole at $(1, -1)$.

$$\left. \frac{2(x-1)}{(x-1)(x-3)} \right|_{x=1} \Rightarrow \left. \frac{2}{x-3} \right|_{x=1} \Rightarrow \frac{2}{(1)-3} \Rightarrow -1$$

Notice also that $x = 3$ causes the denominator to be zero while the numerator is 4 (non-zero). This means there will be a vertical asymptote at $x = 3$.

When confirming graphically (shown at the right), the vertical asymptote is clearly visible but the hole is usually not. To confirm there is a hole at $x = 1$, use the “trace” feature on your grapher to verify the function is undefined.



End Behavior, Including Horizontal and Slant Asymptotes

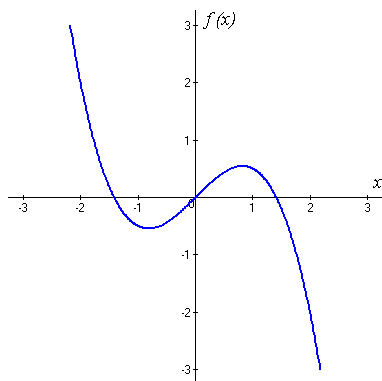
End behavior describes a function's tendencies as it continues forever in either the positive and negative direction. Behavior in the negative direction is referred to as left end behavior while right end behavior describes the positive direction.

A type of limit notation from calculus is used to describe end behavior. The first piece of the notation describes the direction (left or right) and the second piece describes the function's behavior in that direction. For example, the notation $x \rightarrow -\infty, f(x) \rightarrow 4$ is read "as x approaches negative infinity, the function $f(x)$ approaches 4" and means that the function values are leveling off at $y = 4$ as x becomes more and more negative.

In most cases, end behavior will be unbounded, that is $f(x) \rightarrow \pm\infty$, or it can level off to a particular constant (there are other possibilities when considering non-algebraic functions). Left and right end behaviors should be described separately since they are not always the same.

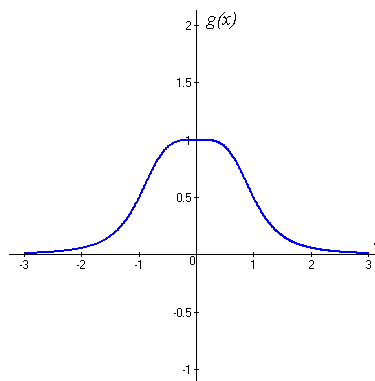
Example

Consider the functions $f(x)$ and $g(x)$ shown below.



Left: $x \rightarrow -\infty, f(x) \rightarrow \infty$

Right: $x \rightarrow \infty, f(x) \rightarrow -\infty$



Left: $x \rightarrow -\infty, g(x) \rightarrow 0$

Right: $x \rightarrow \infty, g(x) \rightarrow 0$

It is often advantageous to describe the end behavior as an asymptote, when possible. When the end behavior is a constant, for example, there is a horizontal asymptote in the graph of the function and it can be identified by the linear equation, $y = b$.

Some unbounded end behavior can also be described as a slant (linear) asymptote or polynomial (non-linear) asymptote. To find the equation for such an asymptote – a task generally reserved for rational functions – the function needs to be rewritten using polynomial division. After the division, the result will have a quotient (called $q(x)$, a polynomial of some degree) and remainder. The equation for the end behavior asymptote will be $y = q(x)$.

End Behavior, Including Horizontal and Slant Asymptotes (continued)

For rational functions, there is a shortcut for determining the end behavior based on the degree in the numerator and denominator:

- If the degree in the numerator is less than the degree in the denominator, there will be a horizontal asymptote at $y = 0$.
- If the degree in the numerator is greater than the degree in the denominator, there will be a slant or polynomial asymptote (and not a horizontal asymptote).
- If the degree in the numerator is equal to the degree in the denominator, there will be a horizontal asymptote at $y = k$, where k is the ratio of the lead coefficient in the numerator over the lead coefficient in the denominator.

Finally, it is important to note that unlike most vertical asymptotes, a function can cross its end behavior asymptote.

Example

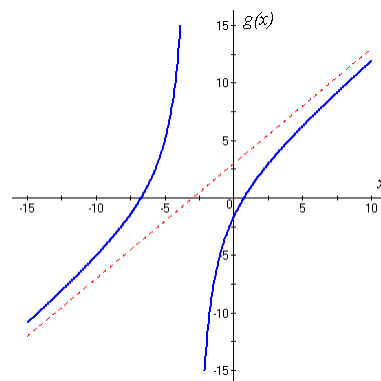
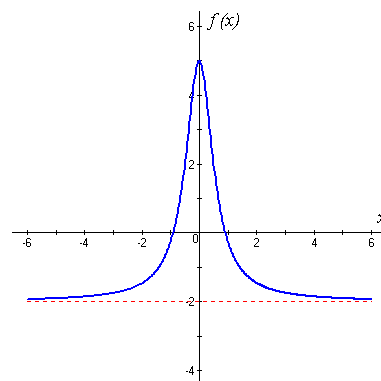
Determine the end behavior asymptotes for the functions $f(x) = \frac{5 - 6x^2}{3x^2 + 1}$ and $g(x) = \frac{x^2 + 6x - 5}{x + 3}$.

$f(x)$ is a rational function with equal degrees in the numerator and denominator (both are 2). This means there will be a horizontal asymptote at $y = k$, where k is the ratio of the lead coefficient in the numerator (-6) over the lead coefficient in the denominator (3). Therefore, there is a horizontal asymptote at $y = -6/3 = -2$.

$g(x)$ is also rational, but this time the degree in the numerator is greater than the denominator's. This means there will be a slant asymptote (since the difference in degrees is 1, the polynomial asymptote will be of degree 1). To find this asymptote's equation, the expression needs to be rewritten using either synthetic or long division (steps not shown here).

$$\frac{x^2 + 6x - 5}{x + 3} \Rightarrow x + 3 + \frac{-14}{x + 3}$$

The quotient is $x + 3$, so the slant asymptote will have the equation $y = x + 3$.



Extreme Values

An extreme value in a function can be thought of as a maximum or minimum function value, but there are two unrelated types of extreme values (often called extrema). The first is called an absolute or global extreme value, the second a relative or local extreme value.

By definition, the absolute maximum is the greatest function value the function takes on its domain. Graphically, this is the highest y -value on the graph of the function. A function cannot have more than one absolute maximum, but that value can occur at several different places in the domain.

Similarly, the absolute minimum is the least function value on the domain and is represented graphically as the lowest y -value on the graph.

A relative maximum is the greatest function value on the graph in a small area. Think of it as the point at the top of a little “hill” in the graph. It does not have to be the greatest value for the entire function, just when compared to domain values to the immediate left and right.

A relative minimum is the least function value on the graph in a small area. Think of it as the point at the bottom of a little “valley” in the graph. It does not have to be the least value for the entire function, just when compared to domain values to the immediate left and right.

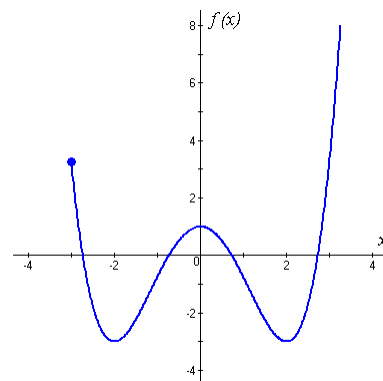
A function can have any combination of absolute and/or relative extrema, or it might have none at all. Some values can be considered both absolute and relative. It is key to understand that absolute and relative extrema are completely unrelated to each other. Also note that endpoints cannot be considered relative extreme values since there are no function values on both sides.

Locating extrema algebraically generally requires calculus techniques, which are not discussed in this summary. They can, however, be approximated use the “maximum” or “minimum” commands in the graphing calculator. Although extrema are function values, it is common for them to be stated as coordinates, (a, b) , as well as using function notation, $f(a) = b$.

Example

In the function $f(x)$ shown in the graph at the right, there is no absolute maximum since the graph extends upward on the right side (since the graph is unbounded, there is no highest point). There is, however, an absolute minimum value of -3 , identified by the points $(-2, -3)$ and $(2, -3)$.

There is a relative maximum at $(0, 1)$ and two relative minimums at $(-2, -3)$ and $(2, -3)$.



Intervals of Increasing and Decreasing Behavior

A graph is increasing if it has a positive slope (it is going “uphill” as you read left-to-right) and decreasing if it has a negative slope (it is going “downhill”). Most functions have a combination of these two behaviors, so the domain is broken up into subintervals to identify for what x -values the function is increasing and what x -values it is decreasing.

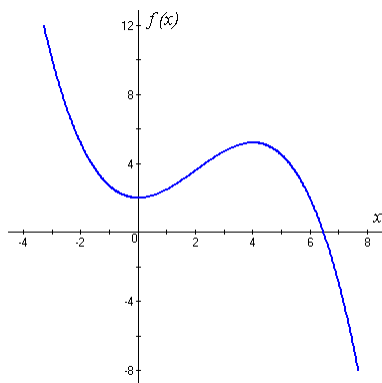
To find the intervals, start at the left of the graph and determine if it is increasing or decreasing. Move to the right until the graph changes its direction or there is a “break” in the graph (such as with a vertical asymptote). State the interval from the beginning to the x -value where the change occurred. Repeat this technique until another change happens. Continue the process until you reach the end of the graph.

In most situations, increasing and decreasing behavior will only change if there is a relative extreme value or a vertical asymptote. Knowing this in advance can help in identifying where to break up the domain subintervals.

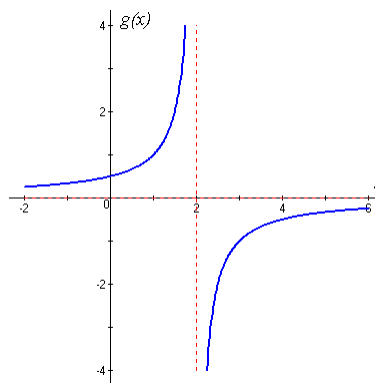
Note that if a graph is neither increasing nor decreasing, it will be constant. While this does not occur frequently, you would state the domain subinterval for this behavior as well.

Example

Consider the functions $f(x)$ and $g(x)$ shown below.



Increasing: $(0, 4)$
Decreasing: $(-\infty, 0) \cup (4, \infty)$



Increasing: $(-\infty, 2) \cup (2, \infty)$
Decreasing: *never*

Reading from left-to-right, the function $f(x)$ is initially decreasing, then increasing, and then decreasing. The behavior changes first at the relative minimum where $x = 0$ and then again at the relative maximum where $x = 4$.

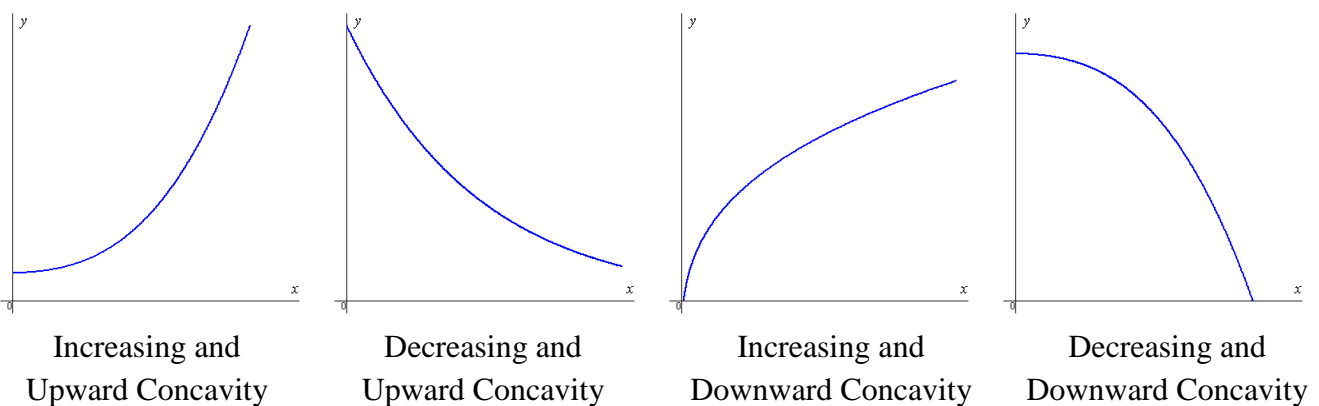
The function $g(x)$ is always increasing on both sides of the vertical asymptote at $x = 2$. Because that value is not in the domain of the function, the interval of increase must be broken into two subintervals at that location.

Concavity and Points of Inflection

Concavity in a graph is used to describe how a function is increasing or decreasing. Although a formal definition of concavity requires calculus, its idea can be described by the following:

- A function has upward concavity if it is increasing more or decreasing less as x increases. Its graph will take on the shape of \cup .
- A function has downward concavity if it is increasing less or decreasing more as x increases. Its graph will take on the shape of \cap .
- A function has no concavity if its increasing or decreasing behavior is constant. Its graph will be linear.

Below are graphical representations of curves with a variety of increasing and decreasing behaviors with upward and downward concavity.



While some graphs, such as parabolas, have the same concavity for all values of x , others have varying concavity throughout the domain. Just as with increasing and decreasing behavior, it is typical to describe concavity using subintervals.

The coordinate on the graph of a function where its concavity changes is called a point of inflection, but calculus is usually required to find the exact location of a function's inflection point. Concavity can also change when there is a vertical asymptote, although this is not considered a point of inflection (since there is no point on the graph).

Example

Consider the function $f(x)$ graphed at the right. Its concavity changes at the vertical asymptotes and also near $(0, 0)$, which means the origin is a point of inflection. The concavity can be described using interval notation.

- Upward Concavity: $(-2, 0) \cup (2, \infty)$
- Downward Concavity: $(-\infty, -2) \cup (0, 2)$

